The Impact of Cooperative Guided Reflection on Student Learning: The Case of Optimization Problem Solving in Calculus I

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Introduction and Background

One of the major goals and challenges of undergraduate mathematics programs is to develop students' problem solving skills. The Calculus I unit on optimization problems is an important undergraduate mathematics topic that focuses on the problem solving aspect of mathematics. There is a large and growing literature on the value of both writing and cooperation in the learning of mathematics (Meier et al, 1998; Morgan, 2002; Reynolds et al 1995; Sterret 1992). I developed an assignment for Calculus I students that synthesizes the mathematical goal of problem solving with the pedagogical goals of writing to learn and cooperative learning. In the assignment students cooperatively solve optimization problems and then use a list of prompting questions to guide them in reflection on their problem solving process. Next, the students work together to write both their mathematical work and their learning reflections. Assignment guidelines are given in Appendix I. The cooperative guided reflection (CGR) assignment may be incorporated in many mathematics courses to aid student learning in other important course topics. Furthermore, it can be incorporated into a wide variety of course delivery methods.

We will address the impact of the CGR assignment on student learning about problem solving. How do the collaborative, writing and reflective aspects of the assignment effect student learning? What is the impact of the assignment on attitudes and beliefs about mathematical problem solving? How does exam performance on optimization problem solving compare between students who are assigned the project and students who are not assigned the project? Does improved knowledge of optimization problem solving extend to other mathematical problem solving?

Research Methodologies

The impact of the CGR assignment was evaluated using three methodologies. Pre and post surveys of student understanding of problem solving concepts and attitudes about problem solving were administered and analyzed. A comparison of exam performance on optimization problems between students who do the assignment and students in a different

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section of Calculus I who do not do the assignment was carried out. Finally, the students' written reflections in the assignment were analyzed for evidence of growth in student learning.

The subjects in the study were students with a wide variety of majors enrolled at a Midwestern university with an enrollment of 6,000 students with an average ACT composite score of entering freshman at 22.0, compared to a national average of 21.1 for that year. The study took place in the "off-semester:" most of the students required remediation in algebra and trigonometry before enrolling in Calculus I and several of the students were re-taking Calculus I. The students had completed a different CGR assignment for another topic earlier in the semester.

Research Frameworks

We used four frameworks to guide the analysis: The Rule of Four for approaching calculus: verbal, graphical, algebraic, and numerical; George Polya's Problem Solving Heuristic: understand the problem, think of a plan, carry out the plan, and look back; Alan Schoenfeld's Framework for the Analysis of Mathematical Behavior: resources, heuristics, control, belief systems; and R. Shavelson, C. Bennet and J. Dewar's Taxonomy for Mathematical Knowledge-Expertise: interest, confidence, factual, procedural, schematic, strategic, epistemic, and social. Each of these frameworks is useful in different ways to the analysis of student understanding and growth in optimization problem solving skills. We use the lens of the Rule of Four and Polya's heuristic to guide our analysis of students' growth in calculus problem solving. Examining student learning through Schoenfelds' framework illuminates student behaviors as they collaborate and write about their problem solving processes. Investigating student work through the lens of Shavelson, Bennet and Dewar's Taxonomy helps in the analysis of student attitudes and beliefs.

Research Findings

The comparison of exam scores between students doing the CGR assignment and the control group showed no statistically significant difference. However CGR was associated with better learning outcomes regarding the issues of domain and testing critical numbers when working optimization problems.

The students' written work and the pre-post surveys revealed several associations between positive learning outcomes and CGR. There is an association between CGR and a growth in understanding of the terms verbal, algebraic, graphical and numerical thinking. CGR is associated with a growth in student value and comprehension of the Polya heuristic "understand the problem." Students transferred knowledge of optimization problem solving to algebra problem solving, especially transferring their symbolic and visual thinking. Student work demonstrated that this knowledge transfer is associated with CGR. Two student belief systems showed growth that was associated with CGR: the amount of time it takes to solve a mathematics word problem and the value of cooperative work in mathematics. CGR is associated with a growth in attitude about the applicability of calculus. Several other learning outcomes were observed through analysis of the student work in the CGR assignment. Students demonstrated growth in resources, heuristic and control for problem solving.

Pre and Post Surveys

Before beginning the unit on optimization problem solving students responded to a three-part survey. The first part of the survey, item #1, was a word problem that required only basic geometry and algebra for its solution. The second part of the survey, item #2, was a standard optimization problem. With both items, students were asked to write down ideas about how to solve the problem, to attempt to give a complete solution and to determine whether possible given answers were plausible. The third part was a survey of student attitudes and beliefs about mathematics problem solving. (See Appendix II for the complete survey.)

Six weeks after the pre-survey, students responded to a similar post-survey. There are a few key differences between the pre and post surveys. Because students had been so successful with determining the plausibility of given answers, this part was not included in the post-survey. The optimization item on the post-survey was assigned as a quiz item, rather than solely as an item they were completing for the research project. There was an extra item on the post-survey about students' attitudes towards the value of the cooperative guided reflection assignment. Forty-nine students participated in both surveys.

Item #1: Algebra Word Problem Pre-Post Survey Analysis

The following algebra item appeared on the pre and post surveys:

> A box with a square base and an open top will be constructed from 689 square centimeters of material. What should be the dimensions of the base if the height of the box is to be 10 centimeters?

On the pre-survey students were unsuccessful at solving this item: on a 6-point scale 78% of the students earned 0-1 points; no student was able to solve it completely. However, many students had good strategies for approaching the item: 82% of the students used an appropriate visual aid; 16% of the students drew a correctly labeled visual aid. Furthermore, the students were quite successful at Polya's "look back" heuristic. As figure 1 demonstrates, 74% of the students were able to determine whether at least three of the four possible given answers were correct and to give appropriate reasons for their answers.

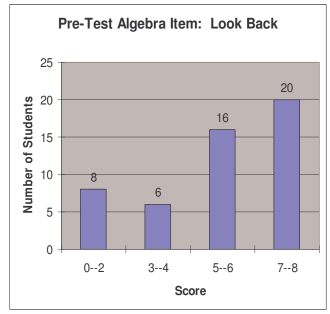

Figure 1

Student responses were analyzed using five different rubrics. One rubric was a simple 6-point scale for solving the item. Figure 2 shows the scores in the pre-survey and post-survey using this rubric. While the students had received no direct instruction on this kind of problem and students understood that their performance on this problem was unrelated to their course grade, many students improved in their ability to solve it. The percentage of students scoring 0-1 decreased from 78% to 37%; the percentage of students scoring 5-6 points increased from 2% to 35%. When we considered the change in score, student-by-student, we found that 71% of the class improved their score.

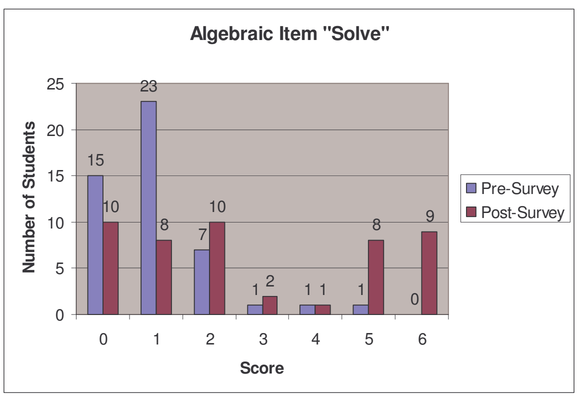

Figure 2

The other four rubrics used to analyze student work on this item were verbal, geometrical, algebraic, and numerical rubrics. Each of these rubrics focused solely on students' ability to use each of these four types of thinking. There was very little change in students' verbal or numerical thinking of this item. There was marked improvement in their quality of geometrical and algebraic thinking.

Figure 3 shows that students made clear progress in their geometric approach to this item. Although 18% of students used no visual aid on the pre-test, every student used a relevant visual aid on the post-test. Furthermore, the number of students with an appropriately labeled visual aid increased from 16% to 57%. When we consider the change in use of geometrical thinking on a student-by-student basis, we found that 69% of the students showed advancement.

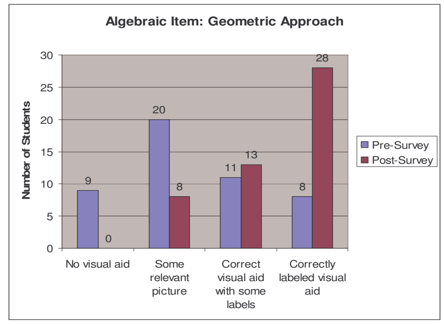

Figure 3

Similarly students made progress in their quality of algebraic thinking. The number of students who used no algebraic symbols decreased from 45% to 10%. The number of students who showed a complete algebraic solution increased from 2% to 20%. On a student-by-student basis, we found that 65% of the students improved their use of algebra to solve this problem.

Before instruction many students had skills in the first and last steps in Polya's problem solving heuristic: "understand the problem" and "look back." After instruction in calculus optimization problem solving, students' ability in solving this algebra word problem improved overall. In particular students made advances in their geometrical and symbolic approaches to this item. Students were able to transfer their knowledge of problem solving in a calculus optimization problem setting to problem solving in an algebra setting.

Does CGR play a role in this transfer of knowledge or was the transfer caused solely by classroom instruction? When we analyzed the student work we found evidence that demonstrated students' growth in thinking visually and symbolically.

> "This problem taught me a lot about labeling the picture to make it easier to work out. When I labeled it the easy way, finding the derivative was much easier."
> "You have to get the $(x - 12)$ and the $(y - 8)$ before you can even start the problem."

We see that students are getting a deeper understanding of what it means to think visually and symbolically and they are recognizing the need to give their attention to these kinds of thinking. Furthermore, student writing shows that students use their broader knowledge to transfer ideas in problem solving. "Since I received help on number 16, I didn't need help on this one, being as the concepts were the same."

Item #2: Calculus Optimization Problem Pre-Post Survey Analysis

The following optimization item appeared on the pre-surveys and as a quiz item post instruction:

> Find the area of the largest rectangle that can be inscribed in a right triangle with legs of length 6 centimeters and 10 centimeters, if two of the sides of the rectangle lie on the legs.

A very similar item appeared in the CGR assignment. This particular item was chosen because in previous semesters, students had written both that this was the most difficult problem and that it was the easiest problem to solve.

As with item #1, on the pre-survey, students demonstrated skills in using visual aids to "understand the problem" and skills in "looking back" to determine whether given answers were plausible. Although the pre-surveys did not specifically ask students to use a visual aid, 100% of the students did so; and 82% of the students had a correct visual aid with some appropriate labels. On the pre-surveys 59% of the students were able to determine whether at least three of the four possible given answers were correct and to give appropriate reasons for their answers.

On the pre and post surveys item #2 was analyzed using five different rubrics, similar to the ones used for item #1: overall solution, verbal, geometrical, symbolic and numerical thinking. Students made improvements in their work using all five rubrics. These improvements can also be connected to the CGR assignment. A confounding factor is that students may have tried harder to demonstrate complete understanding on the post-quiz than on the pre-survey since the post-quiz was part of their course grade and the pre-survey was not. However, students were given ample time on the pre-survey and most students seemed genuinely interested in helping the research process, so it is unlikely that this is a major factor.

On the simple 8-point scale for overall solution of the item, 100% of the students scored 0-1 points on the pre-survey (as expected) while 47% of the class scored 5-8 points on the post-quiz. While the post-quiz performance seemed discouraging, the analysis of the work with the other rubrics indicated significant student learning.

The verbal rubric is skewed towards the pre-survey where students were asked to "write down ideas." Nonetheless, 53% of the students used words more and more appropriately on post-quiz than on the pre-survey. While 10% of the students wrote an incomplete verbal description of a solution process on the pre-survey (and none of the students wrote a complete verbal description), 29% of the students wrote either an incomplete or a complete verbal description on the post-quiz (and half of those students who wrote description wrote a complete verbal description).

The results on the verbal rubric are striking because they indicate that using words in the CGR assignment may transfer to the use of words as an aid to problem solving in a quiz setting. This is supported by the students' writing.

> "This problem involved a lot of numerical thinking, and as with most of the problems, verbal thinking as a group."
> "Richard explained to us that we needed to then take those numbers and place them into the equations of the sides. We did a lot of our thinking verbally for this problem."

These students are learning to value the use of words to help them think about mathematics. They are also internalizing the idea of verbal thinking in mathematics.

Our numerical rubric may also be skewed towards the pre-survey, since students without instruction in calculus problem solving may naturally revert to the concrete process of numerical thinking when approaching an optimization problem for the first time. In fact, 12% of the students tried some numbers (that is they found an area for a particular rectangle) on the pre-survey but did not try this approach on the post-quiz, instead directly using the more abstract symbolic approach. While 47% of the students had no change in their level of numerical approach to this item, 41% of the students improved in their level and quality of numerical approach, either trying a set of numbers on the post-quiz when they had tried none on the pre-survey, or trying more sets of numbers on the post-quiz than they had on the pre-survey.

Analysis using the geometric rubric demonstrated remarkable student growth. While 4% of students had an appropriately labeled visual aid on the pre-survey, 90% of the students had one on the post-quiz. Furthermore, even though students demonstrated skill in the use of visual aids on the pre-survey, when we considered the data student-by-student, 90% of the students made improvements in their use of a visual aid in approaching this item, while the other 10% showed no change in their use of a visual aid. The combination of classroom instruction and the CGR assignment clearly helped students advance in their use of geometrical thinking for problem solving.

There was also a dramatic increase in the quality of algebraic thinking in the solution of the optimization problem. In figure 4 we see that most students did not use any algebraic symbols on the pre-survey, and of the few that did only 12% had a correct algebraic formula. In contrast, on the post-quiz 100% of the students used some algebraic symbols and 69% of the students wrote the two correct algebraic formulas needed to solve this problem with calculus. When we considered the data on a student-by-student basis, we found that 98% of the students made improvements in their algebraic thinking on this item.

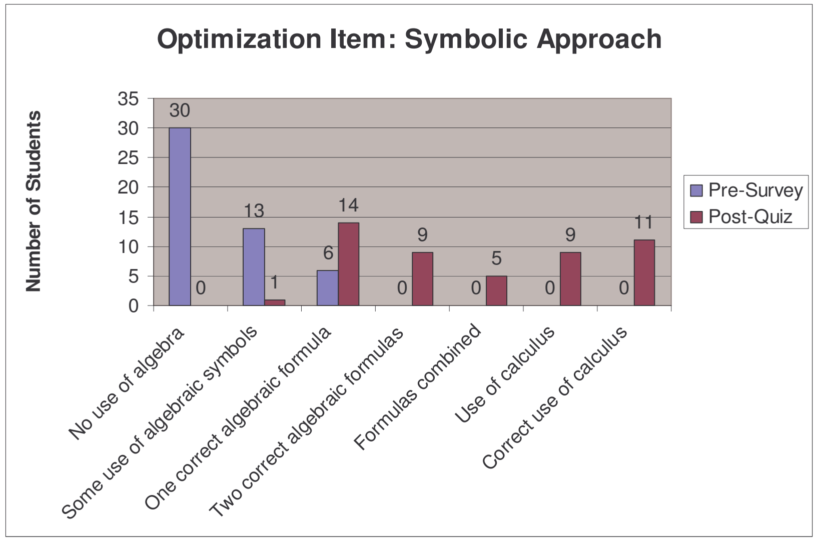

Figure 4

One may conjecture that the growth in all four approaches to calculus optimization problems that were observed in the pre-post surveys, was caused by classroom instruction, that CGR was not important. However, a clear theme that emerges from students' written work is that through writing, students are getting a deeper understanding of what it means to think algebraically, graphically, numerically and verbally. They are separating these ideas; they are attaching the words to the ideas; and they are growing in their understanding of the ideas.

The CGR assignment engages students in metacognition about ways to approach calculus ideas. Many researchers have contributed to an understanding of the importance of metacognition in mathematics education (Schoenfeld, 1985; Pugalee 2001; Bransford, Brown and Cocking, 2000). CGR is a practical way to implement metacognition in the university mathematics classroom setting.

Student Attitudes and Beliefs Pre-Post Survey Analysis

In addition to cognitive components, students' mathematical behavior is affected by both affective components (Bennet & Dewar, 2007) and belief systems (Schoenfeld, 1985).

Affective components include students' level of interest and confidence in calculus optimization problem solving. Belief systems refer to students' beliefs about what kinds of mathematical behavior will be of value in their work on calculus optimization problem solving. The third part of the pre-post survey was designed to investigate how these components of mathematical behavior may change in the presence of CGR.

The questions asked on this part of the survey directed towards affective components included items about creativity, confidence, and enjoyment, in mathematics problem solving and about and the applicability of calculus. Belief systems were addressed with items about perceived time to solve mathematics word problems; the value of collaboration in learning mathematics; and with items about knowledge of using verbal, graphical, algebraic and numerical thinking in mathematics, terms which had been used in class for five weeks.

Overall this group of students was neutral to positive in their attitudes and beliefs about mathematical problem solving; every question received between 75% and 100% positive or neutral answers on both pre and post survey. Although there was not a large change in attitudes in this six-week period, three of the items on both the pre-survey and post-survey showed interesting results.

The affective component that showed an interesting change was students' attitude about the applicability of calculus. Although most of the class showed no change in attitude on this issue, indeed most students were very positive about this question even on the pre-survey, figure 5 shows that some students (33%) expanded their view of the applicability of calculus. A view towards greater applicability of calculus implies a higher level of interest in mathematics, which is connected with better mathematics problem solving success.

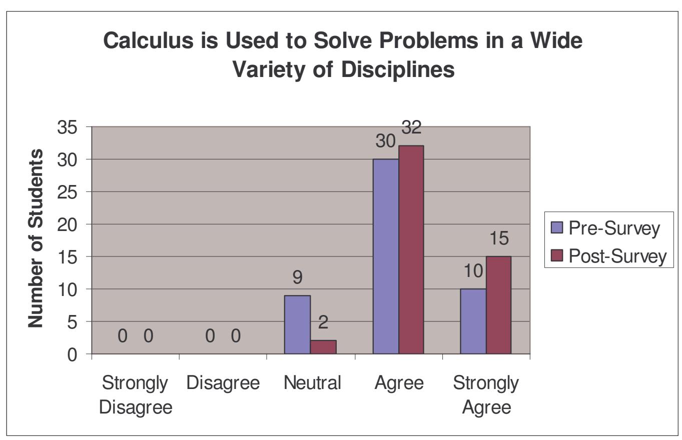

Figure 5

The CGR assignment shaped students' attitudes about the applicability of mathematics. The student writing indicated that through CGR students were making connections between calculus and the "real world;" making connections between calculus and their own interests; and creating their own examples of how to apply calculus. A typical student reflection on applications is in reference to a problem about minimizing cost to build a box.

> "I think that these types of problems really relate to my field of study (geology and biology). I'll probably be working for a consulting firm where minimizing cost will be a large issue."

Other students wrote about how applications of calculus make mathematics easier to understand by connecting it to real life. Some students even suggested how to extend a calculus application in one setting to a distinctly different setting.

There was in increase in students' estimates of the amount of time they use to solve a mathematics word problem. Figure 6 shows that the percentage of students estimating ten minutes or less decreased from 53% to 37%, while the percentage of students estimating ten minutes or more increased from 45% to 61%. When we considered data student-by-student, we found that although 55% of the students made no change in their estimate of time to solve a word problem, 35% of the students increased their time estimates. The implication is that if students believe that they will need more time to solve a mathematics word problem, they may have more success, simply because they are expecting to think about a problem for a longer time period. A maturity in mathematical belief systems is connected to more successful problem solving (Schoenfeld, 1995).

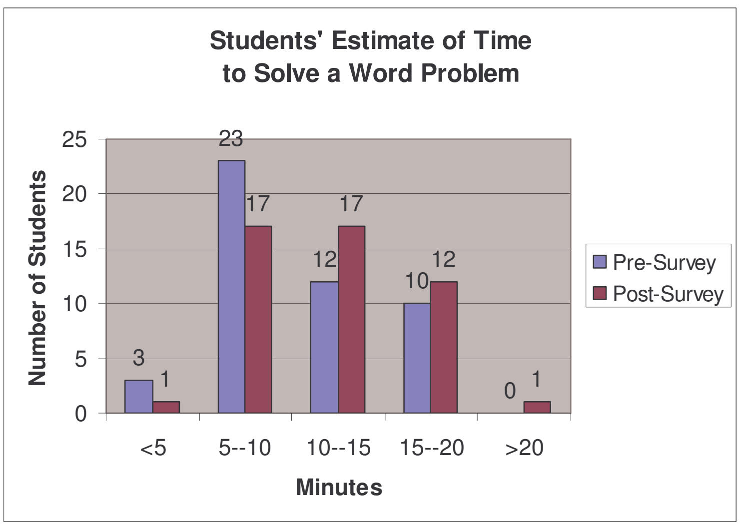

Figure 6

In the CGR assignment, some students reflected on how much time they needed to solve mathematics word problems, even though the issue of time was not addressed directly in the assignment. The student writing demonstrates that they are using CGR to advance their own

mathematical belief systems about this matter. One student wrote: "This problem taught me the most but it also took the largest and hardest amount of time to complete."
Another student demonstrated that increasing expectation of time to solve a problem was an important factor in his ability to correctly complete the solution. "I have come to believe that of the problems I do, there are no hard problems, just long problems."

There was also a small improvement in student attitudes about the value of cooperation in learning mathematics as shown in figure 7. When students took the pre-survey they were working on a CGR assignment for combining differentiation rules and had done some other work collaboratively in class for five weeks. So they were already quite familiar with several forms of cooperative work in mathematics. When we studied the data student-by-student, we found that most of the class (57%) did not change their view on cooperative work, however 33% of the class did raise their level of agreement with the statement "working with classmates helps me to learn mathematics." Reynolds, Hagelgans, Schwingendorf et al (1995) report that students who learn cooperatively learn calculus "more quickly and more thoroughly." (p.18) According to Leamnson (1999), in order to benefit from cooperative learning, students "must become convinced that it is worthwhile." (p. 79) If students change their belief systems to so that they value cooperative learning of mathematics more highly, then we can expect them to experience greater success at mathematical problem solving.

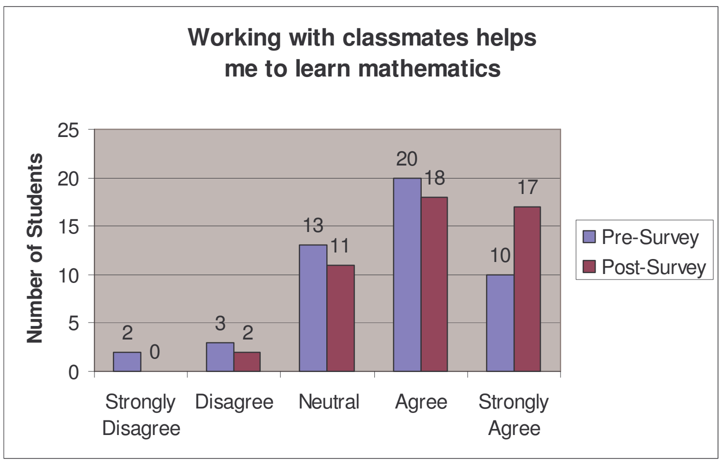

Figure 7

Through CGR students reflect on the value of working with classmates to learn mathematics. They write about how they use oral communication with each other to help them work through exercises; about enjoying working together; and about depending on one another to learn. For example:

> "We solved this by going over every step and talking through it."
> "It was fun because we worked very well together to get the answer."

> "Only Max in our group had this one done and I relied very much on him to help me get through this one."

The post-survey included an item not appropriate for the pre-survey about students' attitudes towards the value CGR. The majority of the students (51%) did not believe that the project affected their attitudes and beliefs about mathematical problem solving. Students are making an accurate assessment as most of their answers to this part of the survey did not change. However 33% of the students did feel that the project changed their attitudes and beliefs. So for a significant number of students, the small changes in attitudes and beliefs discussed above seem to be correlated with completing the cooperative guided reflection assignment.

Conclusions of Pre and Post Surveys

The pre-surveys demonstrate that students begin the optimization unit with knowledge about the importance of visualizing a problem and of ways to draw pictures that facilitate this thinking. They also begin with knowledge of how to "look back" and determine if their answer is reasonable. However, no student could completely solve a simple optimization problem, and the vast majority of students could not solve a "word problem" that required only algebra in its solution. The pre-surveys also reveal that the students were neutral to positive in their attitudes about problem solving.

The post-surveys showed students' improvement in their ability to solve problems posed through verbal descriptions. Although very few students could solve the pre-survey word problem that involved only algebra before the unit on optimization in class, most of the students could do so afterwards, even though there was no instruction for solving this type of algebra problem. Students especially made advances in their geometric and symbolic thinking. Students transferred knowledge of optimization problem solving to algebra problem solving.

Not surprisingly, post instruction students had much more success at solving the optimization item. Students made dramatic advances symbolic and geometrical thinking, and more modest advances in verbal and numerical thinking.

There was very little change in attitudes and beliefs during the six-week period between pre and post survey. Those attitudes and beliefs that did change concerned the applicability of calculus, estimate of time to solve a problem, and the value of cooperation in learning mathematics. Students saw calculus as more widely applicable, estimated a longer period of time to solve word problems, and put higher value on working with classmates in the learning of mathematics.

Other Positive Outcomes of CGR Indicated by Student Work

As we examined the reflections of students we discovered that through CGR students are gaining greater depth and breadth to their repertoire of resources. Students wrote:

> "I didn't really remember how to the distance formula before we got these problems. Now, it seems easy because it is the Pythagorean Theorem."
> "What I learned about problem solving is to remember there can be two values for a maximum, which I would have forgotten if I hadn't drawn a diagram."
> "It is important to remember that the function to optimize can be simplified before finding its derivative. For example, we can find the derivative of d^2 rather than d."

We see that they are gaining a greater depth of understanding of previously learned mathematics (such as the distance formula), greater depth of understanding of mathematics introduced in the calculus course (such as the maximum of a function) and greater breadth of resources to use to solve problems (such as maximizing the square of a function).

Another observation we made is that students use CGR to improve their understanding and use of Polya's problem solving heuristic, especially the steps "understand the problem" and "look back." Students wrote:

> "… the hardest part is to read the problem and understand it."
> "Looking back helped us all realize that our first domain didn't makes sense and it helped us work together to get the right answer."

We see students internalizing the importance of both understanding the problem at the beginning of the problem solving process and examining their final answer when they have finished the problem.

A final observation is that students use CGR to help them become more expert at decision making in problem solving or what Shoenfeld (1985) terms "control". One student wrote, "The challenging part of this problem was trying to get the right equations to solve the problem." This student is recognizing the need to make a decision about what formulas to use and how to combine the formulas prior to invoking calculus methods. Another student wrote, "For me it is harder to solve optimization problems because you have to draw your picture, label the variables, come up with your own equation and hope that all the steps you did lead up to the right answer and the method changes for each problem." This student is recognizing the need to make careful decisions as she must create her own equation and as she is aware that her method for each problem will be different. Both of these students are using writing to deepen their awareness of the role of decision making in problem solving.

While these gains in resources, heuristic and control might have been made without the CGR assignment, we do know that the reflection process aids in metacognition which in turn improves success in mathematical problem solving (Candia 2002; Pugalee 2001).

Comparison of Exam Scores with non-CGR students

A comparison of exam performance on optimization problems between students who do the CGR assignment and students in a different section of Calculus I who do not do the CGR assignment was carried out. There were three sections of Calculus I involved in the study: two sections (N=26 and N=25) did the CGR assignment and were instructed by the investigator; the control section (N=16) not receiving the CGR assignment was taught by another instructor. Both instructors used the same textbook, an interactive classroom lecture style and assigned similar homework items. The smaller class size of the control group might have been a confounding factor. Students in smaller classes may perform better on exams than students in larger classes.

All students in the study attempted the following optimization item on an hour exam, with the extra instruction: use the principles of problem solving and the optimization plan we discussed in class to solve this problem.

> A storage space designer wants to fence an area of 216 square meters in a rectangular shape and then divide it in half with a fence parallel to one of the sides of the rectangle. How can she do this so as to minimize the cost of the fence?

This item was very similar to a homework item assigned in all sections of the course. The homework item was in an agricultural context, had slightly more difficult numbers than the exam item, and used US customary units rather than metric units. The function that must be minimized is a non-polynomial, rational function.

The exam item was scored using a 15 point scale, with 10 points awarded for setting the problem-up appropriately, essentially an algebra task; 4 points awarded for finding and testing the critical numbers, a calculus task; and 1 point for an appropriate conclusion. The average score for students in the CGR sections was 9.57, and for the non-CGR section was 8.94. This was not a statistically significant difference (p=.29).

There are two issues in solving this item that are emphasized at varying levels when instructors teach optimization in Calculus I. The first is the domain of the function that is to be optimized and the other is using some sort of test to verify that the found critical number is the desired optimum (First Derivative Test, Second Derivative Test, or Closed Interval Test). Both instructors emphasized these issues in classroom discussion; the CGR assignment further highlighted them (see Appendix I). While none of the non-CGR students addressed these issues on the test item, 38% of the CGR students did. Thus, CGR was associated with better learning outcomes regarding the issues of domain and testing critical numbers when working optimization problems.

Student performance on a common final exam optimization item was also compared. The item on the final exam was a more basic optimization item than the item used on the hour exam, involving maximizing a rectangular area with a fixed amount of perimeter material. The function to maximize was a quadratic polynomial. The CGR group scored insignificantly better than the non-CGR group (p=.18).

Student exam performance does not appear to be enhanced by the CGR assignment, except in relation to the minor issues of domain and critical number test. The confounding factor of smaller class size may indicate that CGR students would have performed comparatively better had the non-CGR class size been larger. Importantly, student exam performance is not diminished by the CGR assignment. Requiring students to focus on a supplementary task and attendant ideas did not impair exam performance.

Conclusions

This study showed that cooperative guided reflection is an effective way to incorporate reflection, writing and cooperation the optimization problem solving unit in Calculus I. Students in the study demonstrated growth in resources, heuristic and control for mathematical problem solving. They expanded their ability to think verbally, graphically, algebraically and numerically in solving optimization problem solving; and they increased their awareness of these kinds of thinking. Students transferred their knowledge of optimization problem solving to more basic algebra problem solving. Their mathematical belief systems matured, especially with regard to the applicability of calculus, time expectations for problem solving, and the level at which they value cooperative learning. At the same time, attention to supplemental work and ideas did not impair exam performance and even improved exam performance regarding minor issues of domain and critical number test.

Cooperative guided reflection is a promising practical method to improve student mathematics learning that could be implemented with a wide variety of mathematics topics and in a broad spectrum of course delivery styles. Future research could examine the use of CGR in other settings.

Appendix I—CGR Project Guidelines for Optimization Problem Solving

Using Derivatives to Solve Optimization Problems

1) Work the following problems independently in your homework notebook.
Sec. 4.7 (*from Calculus by Stewart, 5th ed.*) 1, 3, 5, 7, 9, 11, 13, 15, 16, 17, 22, 24, 29, 39
With each exercise, think about whether you chose to use the Candidates Test, 2nd Derivative Test and 1st Derivative Test.
2) Gather as a group to discuss the following ten exercises: 1, 7, 9, 13, 16, 17, 22, 24, 29, 39. You should be prepared for this group discussion by having worked on the problems.
3) Write up a careful solution to the ten exercises: 1, 7, 9, 13, 16, 17, 22, 24, 29, 39.
Make sure that you show all necessary steps. Remember to write down the domain of your function. Pay special attention to checking that your critical number is truly the extremum required. Choose each of the three tests: Candidates Test, 2nd Derivative Test and 1st Derivative Test, at least once in your project.
4) With each item, include a brief discussion of your problem solving process, what you are learning, and how you are collaborating. Here is a list of questions to think about in your discussion.

- a) What did you learn about the problem solving step "understand the problem"? What was challenging or easy about this step?
- b) What did you learn about finding the formula for the function you need to optimize? How did finding this formula differ from other similar problems?
- c) What did you learn about calculating derivatives in solving this problem?
- d) What mistakes did you make in solving this problem? What can you learn from these mistakes?
- e) When you carried out the part of our optimization plan "check that you have truly found a maximum (or minimum)," why did you choose the method (First Derivative Test, Second Derivative Test or Candidates Test) you chose?
- f) How did you help each other solve the problem?
- g) How did you use graphical thinking, numerical thinking, verbal thinking and/or algebraic thinking in the problem?
- h) How do the ideas in the problem relate to problems that arise in your major field of study?
- i) What was the hardest (easiest? most fun? most tedious?) part of solving the problem?
- j) Was this problem more fun than other problems? (Less fun? More difficult? Less difficult?) Why?
- k) What did you learn about the problem solving step "look back" when solving this problem?

Your brief discussion should *not* include a verbal blow-by-blow repeating what you have already stated in your problem solution. You are *not* expected to address each of the eleven items listed above with every item. Instead you are encouraged to think about each item in light of these questions and then give a short synopsis of what you learn about calculus and about problem solving from each item. The focus of your discussion should be what you are learning and how you are working together to learn.
5) Divide up the work of writing the problems up so that everyone does her or his fair share. Do not attempt to put two problems on the same page—each problem will require at least one page.

6) This project is worth 25 points. Usually, everyone in the group will receive the same score. It will be graded on completeness, correctness (which includes following instructions), evidence of collaborative work, and evidence of thinking about the problems. One of the ways I will know you worked together on the problems is that you compared answers and got every answer correct. Another way I will know is that you will tell me some of the discussion about problems that happened when you met.
7) Put the problems in increasing numerical order. Put your names on the first page. Each student should sign the last page indicating that you all contributed to the project. The entire project should then be stapled together. You may want to make photocopies so that you will each have a copy to study for exams. Have fun!

Appendix II—Pre-Survey
Part 1: Consider the following problem:
A box with a square base and an open top will be constructed from 689 square centimeters of material. What should be the dimensions of the base if the height of the box is to be 10 centimeters?
• Write down any ideas you have to begin to solve the problem.
• Write down your best attempt at a complete solution of the problem.
• Here are some possible answers. If you think the answer could be correct, check "plausible;" otherwise, check "incorrect" and explain.

a) 12 cm × 15 cm
_____plausible
_____incorrect Explain:

b) 13 cm^2 × 13 cm^2
_____plausible
_____incorrect Explain:

c) 27 cm × 27 cm
_____plausible
_____incorrect Explain:

d) 13 cm × 13 cm
_____plausible
_____incorrect Explain:

Part 2: Consider the following problem:
Find the area of the largest rectangle that can be inscribed in a right triangle with legs of length 6 centimeters and 10 centimeters, if two of the sides of the rectangle lie on the legs.
• Write down any ideas you have to begin to solve the problem.
• Write down your best attempt at a complete solution of the problem.
• Here are some possible answers. If you think the answer could be correct, check "plausible;" otherwise, check "incorrect" and explain.

a) 30 cm^2
_____plausible
_____incorrect Explain:

b) 3 cm × 5 cm
_____plausible
_____incorrect Explain:

c) 15 cm^2
_____plausible
_____incorrect Explain:

d) 1 cm^2
_____plausible
_____incorrect Explain:

Part 3: Please check the box that most closely corresponds to your attitudes and beliefs about problem solving.

1) Mathematical problem solving is a creative process.
Strongly Agree□ Agree□ Neutral□ Disagree□ Strongly Disagree□

2) Calculus is used to solve problems in a wide variety of disciplines.
Strongly Agree□ Agree□ Neutral□ Disagree□ Strongly Disagree□

3) It usually takes me the following amount of time to solve a word problem in mathematics.
Less than 5 minutes□ 5 to 10 minutes□ 10 to 15 minutes□
15 to 20 minutes□ More than 20 minutes□

4) Working with classmates helps me to learn mathematics.
Strongly Agree□ Agree□ Neutral□ Disagree□ Strongly Disagree□

5) I know how to use numerical thinking when I work mathematics problems.
Strongly Agree□ Agree□ Neutral□ Disagree□ Strongly Disagree□

6) I know how to use verbal thinking when I work mathematics problems.
Strongly Agree□ Agree□ Neutral□ Disagree□ Strongly Disagree□

7) I know how to use graphical thinking when I work mathematics problems.
Strongly Agree□ Agree□ Neutral□ Disagree□ Strongly Disagree□

8) I know how to use algebraic thinking when I work mathematics problems.
Strongly Agree□ Agree□ Neutral□ Disagree□ Strongly Disagree□

9) I use my own creativity when I solve mathematics problems.
Strongly Agree□ Agree□ Neutral□ Disagree□ Strongly Disagree□

10) I am confident in my ability to do mathematical problem solving.
Strongly Agree□ Agree□ Neutral□ Disagree□ Strongly Disagree□

11) I enjoy mathematical problem solving.
Strongly Agree□ Agree□ Neutral□ Disagree□ Strongly Disagree□

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